Challenge problems

1. Analyze continuity of the function

$$f(x) = \begin{cases} \dfrac{1}{q} & \text{if } x \text{ is rational and } x = \dfrac{p}{q} \text{ in lowest terms} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

for x in $[1, 2]$.

2. Consider the function

$$f(x) = \begin{cases} x\cos\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

   (a) Compute the derivative $f'(x)$ for $x \neq 0$.
   (b) Analyze $\lim_{x\to 0} f'(x)$.
   (c) Determine if this function differentiable at 0. If so, find the value of $f'(0)$.

3. Consider the function

$$f(x) = \begin{cases} x^2\cos\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

   (a) Compute the derivative $f'(x)$ for $x \neq 0$.
   (b) Analyze $\lim_{x\to 0} f'(x)$.
   (c) Determine if this function differentiable at 0. If so, find the value of $f'(0)$.

4. Compare your results for 2 and 3. Visualize the difference between the graphs of these two functions near $x = 0$.

5. Consider vector-output functions of the form

$$\vec{r}(t) = A\langle\cos(at), \sin(at)\rangle + B\langle\cos(bt), \sin(bt)\rangle$$

where A, B, a and b are positive constants. From looking at specific cases, we know that the output curve may have "rounded dips", cusps, or "loop-de-loops" depending on the choice of values for A, B, a, and b. Find a condition on these parameters which results in cusps on the output curves.